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Research Article

On the Strong Solution for the 3D Stochastic Leray-Alpha Model

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We prove the existence and uniqueness of strong solution to the stochastic Leray-α equations under appropriate conditions on the data. This is achieved by means of the Galerkin approximation scheme. We also study the asymptotic behaviour of the strong solution as alpha goes to zero. We show that a sequence of strong solutions converges in appropriate topologies to weak solutions of the 3D stochastic Navier-Stokes equations.

1. Introduction

It is computationally expensive to perform reliable direct numerical simulation of the Navier-Stokes equations for high Reynolds number flows due to the wide range of scales of motion that need to be resolved. The use of numerical models allows researchers to simulate turbulent flows using smaller computational resources. In this paper, we study a particular subgrid-scale turbulence model known as the Leray-alpha model (Leray-α).

We are interested in the study of the probabilistic strong solutions of the 3D Leray-alpha equations, subject to space periodic boundary conditions, in the case in which random perturbations appear. To be more precise, let $\mathcal{T} = [0, L]^3, T > 0$, and consider the system

$$\begin{gathered}
d\left(u-\alpha^2 \Delta u\right)+\left[-\nu \Delta\left(u-\alpha^2 \Delta u\right)-u \cdot \nabla\left(u-\alpha^2 \Delta u\right)+\nabla p\right] dt \\
=F(t, u) dt+G(t, u) dW \quad \text{in } (0, T) \times \mathcal{T}, \\
\nabla \cdot u=0 \quad \text{in } (0, T) \times \mathcal{T}, \\
u(t, x) \text{ is periodic} \quad \text{in } x, \int_{\mathcal{T}} u \, dx=0, \\
u(0)=u_0 \quad \text{in } \mathcal{T},
\end{gathered} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ and p are unknown random fields on $[0,T] \times \mathcal{T}$, representing, respectively, the velocity and the pressure, at each point of $[0,T] \times \mathcal{T}$, of an incompressible viscous fluid with constant density filling the domain $\mathcal{T}$. The constant $\nu > 0$ and α represent, respectively, the kinematic viscosity of the fluid and spatial scale at which fluid motion is filtered. The terms $F(t,u)$ and $G(t,u)dW$ are external forces depending eventually on u, where W is an R^m-valued standard Wiener process. Finally, u_0 is a given random initial velocity field.

The deterministic version of (1.1), that is, when $G = 0$, has been the object of intense investigation over the last years. The initial motivation was to find a closure model for the 3D turbulence averaged Reynolds number; for more details, we refer to [1] and the references therein. A key interest in the model is the fact that it serves as a good approximation of the 3D Navier-Stokes equations. It is readily seen that when $\alpha = 0$, the problem reduces to the usual 3D Navier-Stokes equations. Many important results have been obtained in the deterministic case. More precisely, the global wellposedness of weak solutions for the deterministic Leray-alpha equations has been established in [2] and also their relation with Navier-Stokes equations as α approaches zero. The global attractor was constructed in [1, 3].

The addition of white noise driven terms to the basic governing equations for a physical system is natural for both practical and theoretical applications. For example, these stochastically forced terms can be used to account for numerical and empirical uncertainties and thus provide a means to study the robustness of a basic model. Specifically in the context of fluids, complex phenomena related to turbulence may also be produced by stochastic perturbations. For instance, in the recent work of Mikulevicius and Rozovskii [4], such terms are shown to arise from basic physical principals. To the best of our knowledge, there is no systematic work for the 3D stochastic Leray-α model.

In this paper, we will prove the existence and uniqueness of strong solutions to our stochastic Leray-α equations under appropriate conditions on the data, by approximating it by means of the Galerkin method (see Theorem 2.3). Here, the word "strong" means "strong" in the sense of the theory of stochastic differential equations, assuming that the stochastic processes are defined on a complete probability space and the Wiener process is given in advance. Since we consider the strong solution of the stochastic Leray-alpha equations, we do not need to use the techniques considered in the case of weak solutions (see [5–9]). The techniques applied in this paper use in particular the properties of stopping times and some basic convergence principles from functional analysis (see [10–13]). An important result, which cannot be proved in the case of weak solutions, is that the Galerkin approximations converge in mean square to the solution of the stochastic Leray-alpha equations (see Theorem 2.4). We can prove by using the property of higher-order moments for the solution. Moreover, as in the deterministic case [2], we take limits $\alpha \to 0$. We study the behavior of strong solutions as α approaches 0. More precisely, we show that, under this limit, a subsequence of solutions in question converges to a probabilistic weak solutions for the 3D stochastic Navier-Stokes equations (see Theorem 6.5). This is reminiscent of the vanishing viscosity method; see, for instance, [14, 15].

This paper is organized as follows. In Section 2, we formulate the problem and state the first result on the existence and uniqueness of strong solutions for the 3D stochastic Leray-α model. In Section 3, we introduce the Galerkin approximation of our problem and derive crucial a priori estimates for its solutions. Section 4 is devoted to the proof of the existence and uniqueness of strong solutions for the 3D stochastic Leray-α model. In Section 5, We prove the convergence result of Theorem 2.4. In Section 6, we study the asymptotic behavior of the strong solutions for the 3D stochastic Leray-α model as α approaches 0.

2. Statement of the Problem and the First Main Result

Let $\mathcal{T} = [0, L]^3$. We denote by $C^\infty_{\text{per}}(\mathcal{T})^3$ the space of all $\mathcal{T}$-periodic C^∞ vector fields defined on $\mathcal{T}$. We set

$$\mathcal{V} = \left\{ \Phi \in C^\infty_{\text{per}}(\mathcal{T})^3 / \int_{\mathcal{T}} \Phi dx = 0; \nabla \cdot \Phi = 0 \right\}. \tag{2.1}$$

We denote by H and V the closure of the set $\mathcal{V}$ in the spaces $L^2(\mathcal{T})^3$ and $H^1(\mathcal{T})^3$, respectively. Then H is a Hilbert space equipped with the inner product of $L^2(\mathcal{T})^3$. V is Hilbert space equipped with inner product of $H^1(\mathcal{T})^3$. We denote by $(\cdot, \cdot)$ and $|\cdot|$ the inner product and norm in H. The inner product and norm in V are denoted by $((\cdot, \cdot))$ and $\|\cdot\|$, respectively. Let $A = -\mathcal{P}\Delta$ be the Stokes operator with domain $D(A) = H^2(\mathcal{T})^3 \cap V$, where $\mathcal{P} : L^2(\mathcal{T})^3 \to H$ is the Leray projector. A is an isomorphism from V to V'(the dual space of V) with compact inverse, hence A has eigenvalues $\{\lambda_k\}_{k=1}^\infty$, that is, $4\pi^2/L^2 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \to \infty (n \to \infty)$ and corresponding eigenfunctions $\{w_k\}_{k=1}^\infty$ which form an orthonormal basis of H such that $Aw_k = \lambda_k w_k$.

We also have

$$\langle Av, v\rangle_{V'} \ge \beta \|v\|^2 \tag{2.2}$$

for all $v \in V$, where $\beta > 0$ and $\langle \cdot, \cdot \rangle_{V'}$ denotes the duality between V and V'.

Following the notations common in the study of Navier-Stokes equations, we set

$$B(u, v) = \mathcal{P}(u \cdot \nabla)v \quad \forall u, v \in V. \tag{2.3}$$

Then (see [16–18])

$$\langle B(u, v), v\rangle_{V'} = 0 \quad \forall u, v \in V, \tag{2.4}$$

$$\langle B(u, v), w\rangle_{V'} = -\langle B(u, w), v\rangle_{V'} \quad \forall u, v, w \in V, \tag{2.5}$$

$$|(B(u, v), w)| \le C|Au|\|v\||w|, \quad \forall u \in D(A),\ v \in V,\ w \in H, \tag{2.6}$$

$$\left|\langle B(u, v), w\rangle_{D(A)'}\right| \le C|u|\|v\||Aw|, \quad \forall u \in H,\ v \in V,\ w \in D(A), \tag{2.7}$$

$$|\langle B(u, v), w\rangle_{V'}| \le C|u|^{1/4}\|u\|^{3/4}|v|^{1/4}\|v\|^{3/4}\|w\|, \quad \forall u \in V,\ v \in V, w \in V, \tag{2.8}$$

$$|(B(u, v), w)| \le C|u|^{1/4}\|u\|^{3/4}\|v\|^{1/4}|Av|^{3/4}|w|, \quad \forall u \in V,\ v \in D(A), w \in H. \tag{2.9}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\{\mathcal{F}_t\}_{0 \le t \le T}$ an increasing and right-continuous family of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. Let W be a R^m-valued Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \le t \le T}, P)$.

We now introduce some probabilistic evolution spaces.

Let X be a Banach space. For $r, p \geq 1$, we denote by

$$L^p(\Omega, \mathcal{F}, P; L^r(0, T; X)) \tag{2.10}$$

the space of functions $u = u(x, t, \omega)$ with values in Xdefined on $[0, T] \times \Omega$ and such that

(1) u is measurable with respect to (t, ω) and for each t, u is $\mathcal{F}_t$ measurable,

(2) $u \in X$ for almost all (t, ω) and

$$\|u\|_{L^p(\Omega,\mathcal{F},P;L^r(0,T;X))} = \left[E\left(\int_0^T \|u\|_X^r dt\right)^{p/r}\right]^{1/r} < \infty, \tag{2.11}$$

where E denote the mathematical expectation with respect to the probability measure P.

The space $L^p(\Omega, \mathcal{F}, P; L^r(0, T; X))$ so defined is a Banach space.

When $r = \infty$, the norm in $L^p(\Omega, \mathcal{F}, P; L^\infty(0, T; X))$ is given by

$$\|u\|_{L^p(\Omega,\mathcal{F},P;L^\infty(0,T;X))} = \left(E \sup_{0\leq t\leq T} \|u\|_X^p\right)^{1/p}. \tag{2.12}$$

We make precise our assumptions on F and G. We suppose that F and G are measurable Lipschitz mappings from $\Omega \times (0, T) \times H$ into H and from $\Omega \times (0, T) \times H$ into $H^{\otimes m}$, respectively. More exactly, assume that, for all $u, v \in H$, $F(\cdot, u)$ and $G(\cdot, u)$ are $\mathcal{F}_t$-adapted, and $dP \times dt -$ a.e. in $\Omega \times (0, T)$

$$\begin{gathered} |F(t,u) - F(t,v)|_H \leq L_F|u - v|, \\ F(t,0) = 0, \\ |G(t,u) - G(t,v)|_{H^{\otimes m}} \leq L_G|u - v|, \\ G(t,0) = 0. \end{gathered} \tag{2.13}$$

Here $H^{\otimes m}$ is the product of m copies of H.

Finally, we assume that $u_0 \in L^2(\Omega, \mathcal{F}_0, P; D(A))$.

Remark 2.1. The condition 10 is given only to simplify the calculations. It can be omitted; in which case one could use the estimate

$$|F(t,u)|^2 \leq 2L_F^2|u|^2 + 2|F(t,0)|^2 \tag{2.14}$$

that follows from the Lipschitz condition. The same remark applies to G.

Alongside problem (1.1), we will consider the equivalent abstract stochastic evolution equation

$$d\left(u+\alpha^2 Au\right)+\left[\nu A\left(u+\alpha^2 Au\right)+B\left(u,u+\alpha^2 Au\right)\right]\,dt=F(t,u)dt+G(t,u)dW,\qquad u(0)=u_0. \tag{2.15}$$

We now define the concept of strong solution of the problem (2.15) as follows.

Definition 2.2. By a strong solution of problem (2.15), we mean a stochastic process u such that

(1) $u(t)$ is $\mathcal{F}_t$adapted for all $t\in[0,T]$,

(2) $u\in L^p(\Omega,\mathcal{F},P;L^2(0,T;D(A^{3/2})))\cap L^p(\Omega,\mathcal{F},P;L^\infty(0,T,D(A)))$ for all $1\le p<\infty$,

(3) u is weakly continuous with values in $D(A)$,

(4) P-a.s., the following integral equation holds:

$$\begin{aligned}\left(u(t)+\alpha^2Au(t),\Phi\right)&+\nu\int_0^t\left(u(s)+\alpha^2Au(s),A\Phi\right)ds+\int_0^t\left(B\left(u(s),u(s)+\alpha^2Au(s)\right),\Phi\right)ds\\&=\left(u_0+\alpha^2Au_0,\Phi\right)+\int_0^t(F(s,u(s)),\Phi)ds+\int_0^t(G(s,u(s)),\Phi)dW(s)\end{aligned} \tag{2.16}$$

for all $\Phi\in\mathcal{V}$, and $t\in[0,T]$.

Notation 1. In this paper, weak convergence is denoted by $\rightharpoonup$ and strong convergence by $\rightarrow$.

Our first result of this paper is the following.

Theorem 2.3 (existence and uniqueness). *Suppose that the hypotheses* (2.13) *hold, and $u_0\in L^2(\Omega,\mathcal{F}_0,P;D(A))$. Then problem* (2.15) *has a solution in the sense of Definition 2.2. The solution is unique almost surely and has in $D(A)$ almost surely continuous trajectories.*

We also prove that the sequence (u_n) of our Galerkin approximation (see (3.1) below) approximates the solution u of the $3D$ stochastic Leray-α model in mean square.

This is the object of the second result of the paper.

Theorem 2.4 (Convergence results). *Under the hypotheses of Theorem 2.3, the following convergences hold:*

$$E\int_0^t\|u_n(s)-u(s)\|^2_{D(A^{3/2})}ds\longrightarrow 0\quad\text{for } n\longrightarrow\infty,\quad E\|u_n(t)-u(t)\|^2_{D(A)}\longrightarrow 0,\quad n\longrightarrow\infty \tag{2.17}$$

for all $t\in[0,T]$.

Remark 2.5. Theorems 2.3 and 2.4 are also true if one assumes measurable Lipschitz mappings $F : \Omega \times (0,T) \times D(A^{3/2}) \to H$ and $G: \Omega \times (0,T) \times D(A) \to H^{\otimes m}$.

Remark 2.6. For the existence of the pressure, we can use a generalization of the Rham's theorem for processes (see [19, Theorem 4.1, Remark 4.3]). See also [6, page 15].

3. Galerkin Approximations and A Priori Estimates

We now introduce the Galerkin scheme associated to the original equation (2.15) and establish some uniform estimates.

3.1. The Approximate Equation

Let $\{w_j\}_{j=1}^{\infty}$ be an orthonormal basis of H consisting of eigenfunctions of the operator A. Denote $H_n = \mathrm{span}\{w_1, \ldots, w_n\}$ and let P_n be the L^2-orthogonal projection from H onto H_n.

We look for a sequence $u_n(t)$ in H_n solutions of the following initial value problem:

$$\begin{gathered} dv_n + [\nu A v_n + P_n B(u_n, v_n)]dt = P_n F(t, u_n)dt + P_n G(t, u_n)dW, \\ u_n(0) = P_n u_0, \\ v_n = u_n + \alpha^2 A u_n. \end{gathered} \tag{3.1}$$

By the theory of stochastic differential equations (see [20–23]), there is a unique continuous $(\mathcal{F}_t)$-adapted process $u_n(t) \in L^2(\Omega, \mathcal{F}, P; L^2(0,T; H_n))$ of (3.1).

We next establish some uniform estimates on u_n and v_n.

3.2. A Priori Estimates

Throughout this section $C, C_i (i = 1, \ldots)$ denote positive constants independent of n and α.

Lemma 3.1. *u_n and v_n satisfy the following a priori estimates:*

$$\begin{gathered} E \sup_{0 \le s \le T} |v_n(s)|^2 + 4\nu\beta E \int_0^T \|v_n(s)\|^2 ds \le C_1, \\ E \sup_{0 \le s \le T} |u_n(s)|^2 \le C_2, \quad E \sup_{0 \le s \le T} \|u_n(s)\|^2 < \frac{C_3}{2\alpha^2}, \\ E \sup_{0 \le s \le T} |A u_n(s)|^2 \le \frac{C_4}{\alpha^4}, \quad E \int_0^T \|u_n(s)\|^2 ds \le C_5, \\ E \int_0^T |A u_n(s)|^2 ds \le \frac{C_6}{2\alpha^2}, \quad E \int_0^T \left| A^{3/2} u_n(s) \right|^2 ds \le \frac{C_7}{\alpha^4}. \end{gathered} \tag{3.2}$$

Proof. To prove Lemma 3.1, it suffices to establish the first inequality and use the fact that

$$\begin{aligned} |v_n|^2 &= \left|u_n + \alpha^2 A u_n\right|^2 = |u_n|^2 + 2\alpha^2 \|u_n\|^2 + \alpha^4 |A u_n|^2, \\ \|v_n\|^2 &= \|u_n\|^2 + 2\alpha^2 |A u_n|^2 + \alpha^4 \left|A^{3/2} u_n\right|^2. \end{aligned} \tag{3.3}$$

By Ito's formula, we have from (3.1)

$$\begin{aligned} &d|v_n(t)|^2 + 2[\nu\langle A v_n, v_n\rangle_{V'} + \langle B(u_n, v_n), v_n\rangle_{V'}]dt \\ &\quad = \left((2F(t,u_n), v_n) + |P_n G(t,u_n)|^2\right)dt + 2(G(t,u_n), v_n)dW. \end{aligned} \tag{3.4}$$

But then, taking into account (2.4), (2.2) and the fact that

$$\begin{aligned} (F(s,u_n(s)), v_n(s)) &\le C\left(1 + |v_n(s)|^2\right), \\ |P_n G(s,u_n(s))|^2 &\le C\left(1 + |v_n(s)|^2\right), \end{aligned} \tag{3.5}$$

we deduce from (3.4) that

$$\begin{aligned} &|v_n(t)|^2 + 2\nu\beta \int_0^t \|v_n(s)\|^2 ds \\ &\quad \le |v_n(0)|^2 + C_2 T + C_3 \int_0^t |v_n(s)|^2 ds + 2\int_0^t (G(s,u_n(s)), v_n(s))dW(s). \end{aligned} \tag{3.6}$$

For each integer $N > 0$, consider the $\mathcal{F}_t$-stopping time τ_N defined by

$$\tau_N = \inf\left\{t : |v_n(t)|^2 \ge N^2\right\} \wedge T. \tag{3.7}$$

It follows from (3.6) that

$$\begin{aligned} \sup_{s\in[0,t\wedge\tau_N]} |v_n(s)|^2 + 2\nu\beta \int_0^{t\wedge\tau_N} \|v_n(s)\|^2 ds &\le |v_n(0)|^2 + C_8 T + C_9 \int_0^{t\wedge\tau_N} |v_n(s)|^2\, ds \\ &\quad + 2 \sup_{s\in[0,t\wedge\tau_N]} \left|\int_0^s (G(s,u_n(s)), v_n(s))dW(s)\right| \end{aligned} \tag{3.8}$$

for all $t \in (0, T)$ and all $N, n \geq 1$. Taking expectation in (3.8), by Doob's inequality it holds

$$\begin{aligned}
E \sup_{s\in[0,t\wedge\tau_N]} \int_0^s (G(s,u_n(s)), v_n(s)) dW(s) &\leq 3E\left(\int_0^{t\wedge\tau_N} (G(s,u_n(s)), v_n(s))^2 \, ds\right)^{1/2} \\
&\leq 3E\left(\int_0^{t\wedge\tau_N} |G(s,u_n(s))|^2 |v_n(s)|^2 ds\right)^{1/2} \\
&\leq \frac{1}{2} E \sup_{0\leq s\leq t\wedge\tau_N} |v_n(s)|^2 + C_{10}T + C_{11}E\int_0^{t\wedge\tau_N} |v_n(s)|^2 ds.
\end{aligned} \tag{3.9}$$

Next using Gronwall's lemma, it follows that there exists a constant C_1 depending on T, C such that, for all $n \geq 1$

$$E \sup_{0\leq s\leq T} |v_n(s)|^2 + 4\nu\beta E\int_0^T \|v_n(s)\|^2 ds \leq C_1. \tag{3.10}$$

□

The following result is related to the higher integrability of u_n and v_n.

Lemma 3.2. *One has*

$$E \sup_{0\leq s\leq T} |v_n(s)|^p \leq C_p, \quad E \sup_{0\leq s\leq T} |u_n(s)|^p \leq C_p, \tag{3.11}$$

$$E \sup_{0\leq s\leq T} \|u_n(s)\|^p \leq \frac{C_p}{\alpha^p}, \tag{3.12}$$

$$E \sup_{0\leq s\leq T} |u_n(s)|^p_{D(A)} \leq \frac{C_p}{\alpha^{2p}} \tag{3.13}$$

for all $1 \leq p < \infty$.

Proof. By Ito's formula, we have for $4 \leq p < \infty$

$$\begin{aligned}
d|v_n(t)|^{p/2} &= \frac{p}{2} |v_n(t)|^{p/2-2} \\
&\quad \times \left(-\nu\langle Av_n, v_n\rangle_{V'} - \langle B(u_n, v_n), v_n\rangle_{V'} + (F(t,u_n), v_n) + \frac{p-4}{4}\frac{(G(t,u_n), v_n)^2}{|v_n(t)|^2}\right) dt \\
&\quad + \frac{p}{2} |v_n(t)|^{p/2-2} (G(t,u_n), v_n) dW.
\end{aligned} \tag{3.14}$$

Taking into account (2.4) and the fact that

$$|v_n(s)|^{(p/2)-2}(F(t,u_n),v_n) \le C\left(1+|v_n(s)|^{p/2}\right) \text{ (Young's inequality)},$$
$$\frac{(G(s,u_n),v_n)^2}{|v_n(s)|^2} \le C\left(1+|v_n(s)|^2\right), \tag{3.15}$$

we deduce from (3.14) that

$$|v_n(t)|^{p/2} \le |v_n(0)|^{p/2} + C\int_0^t \left(1+|v_n(s)|^{p/2}\right)\, ds + \frac{p}{2}\int_0^t |v_n(s)|^{p/2-2}(G(s,u_n(s)),v_n(s))dW(s). \tag{3.16}$$

Taking the supremum, the square, and the mathematical expectation in (3.16), and owing to the Martingale's inequality it holds

$$\begin{aligned} &E\sup_{0\le s\le T}\left|\int_0^s |v_n(s)|^{p/2-2}(G(s,u_n(s)),v_n(s))\ dW(s)\right|^2 \\ &\le 4E\int_0^T |v_n(s)|^{p-4}(G(s,u_n(s)),v_n(s))^2 ds \\ &\le 4CE\int_0^T (1+|v_n(s)|^p)ds. \end{aligned} \tag{3.17}$$

Applying Gronwall's lemma, it follows that there exists a constant C_p, such that

$$E\sup_{0\le s\le T}|v_n(s)|^p \le C_p \tag{3.18}$$

for all $p \ge 4$. With this being proved for any $p \ge 4$, it is subsequently true for any $1 \le p < \infty$.

Other inequalities are deduced from the relation

$$|v_n(s)|^2 = |u_n(s)|^2 + 2\alpha^2\|u_n(s)\|^2 + \alpha^4|Au_n(s)|^2. \tag{3.19}$$

□

We also have the following.

Lemma 3.3. *One has*

$$E\left(\int_0^T \|v_n(s)\|^2 ds\right)^p \le C_p \quad \textit{for } 1 \le p < \infty. \tag{3.20}$$

Proof. The proof is derived from (4.46), Martingale's inequality, and Lemma 3.2. □

4. Proof of Theorem 2.3

4.1. Existence

With the uniform estimates on the solution of the Galerkin approximations in hand, we proceed to identify a limit u. This stochastic process is shown to satisfy a stochastic partial differential equations (see (4.2)) with unknown terms corresponding to the nonlinear portions of the equation. Next, using the properties of stopping times and some basic convergence principles from functional analysis, we identify the unknown portions.

We will split the proof of the existence into two steps.

4.1.1. Taking Limits in the Finite-Dimensional Equations

Lemma 4.1 (limit system). *Under the hypotheses of Theorem 2.3, there exist adapted processes u, B^*, F^*, and G^* with the regularity,*

$$\begin{gathered}
u \in L^p\left(\Omega, \mathcal{F}, P; L^2\left(0,T; D\left(A^{3/2}\right)\right)\right) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T; D(A))), \\
v \in L^p\left(\Omega, \mathcal{F}, P; L^2(0,T;V)\right), \\
v \in C(0,T;H)\, a.s., \\
u \in C(0,T;D(A))\, a.s., \\
B^* \in L^2\left(\Omega, \mathcal{F}, P; L^2(0,T : V')\right), \\
F^* \in L^2\left(\Omega, \mathcal{F}, P; L^2(0,T;H)\right), \\
G^* \in L^2\left(\Omega, \mathcal{F}, P; L^2(0,T;H^{\otimes m})\right),
\end{gathered} \tag{4.1}$$

such that u, B^*, F^*, and G^* satisfy

$$v(t) + \nu \int_0^t Av(s)ds + \int_0^t B^*(s)ds = v(0) + \int_0^t F^*(s)ds + \int_0^t G^*(s)dW(s) \tag{4.2}$$

where $v(t) = u(t) + \alpha^2 Au(t)$ and $1 \le p < \infty$.

Remark 4.2. We use the following elementary facts regarding weakly convergent sequences in the proof below.

(i) Let S_1 and S_2 be Banach spaces and let $L : S_1 \to S_2$ be a continuous linear operator. If (x_n) is a sequence in S_1 such that $x_n \rightharpoonup x$ (where $x \in S_1$), then $L(x_n) \rightharpoonup L(x)$.

(ii) If S is Banach space and if (x_n) is a sequence from $L^2(\Omega, \mathcal{F}, P; L^2(0,T;S))$, which converges weakly to x in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;S))$, then for $n \to \infty$ the following assertions are true:

$$\int_0^t x_n(s)ds \rightharpoonup \int_0^t x(s)ds,$$
$$\int_0^t x_n(s)\ dW(s) \rightharpoonup \int_0^t x(s)\ dW(s) \tag{4.3}$$

in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;S))$.

Proof of Lemma 4.1 . Using (2.8) and Hölder's inequality, we have

$$E\int_0^T \|P_n B(u_n(t), v_n(t)\|_{V'}^2 \leq C\left(E\sup_{t\in[0,T]}\|u_n(t)\|^4\right)^{1/2}\left(E\left(\int_0^T \|v_n(t)\|^2 dt\right)^2\right)^{1/2}. \tag{4.4}$$

The later quantity is uniformly bounded as a consequence of Lemmas 3.2, 3.3. From (4.4), we can deduce that the sequence $P_n B(u_n, v_n)$ is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;V'))$. On the other hand, from Lemmas 3.1, 3.2, 3.3 and the Lipschitz conditions on F and G, we have that the sequence u_n is bounded in $L^p(\Omega, \mathcal{F}, P; L^2(0,T;D(A^{3/2})) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;D(A))$, the sequence v_n is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;V)) \cap L^2(\Omega, \mathcal{F}, P; L^\infty(0,T;H))$, the sequence $v_n(0)$ is bounded in $L^2(\Omega, \mathcal{F}_0, P; H)$, the sequence $u_n(0)$ is bounded in $L^2(\Omega, \mathcal{F}_0, P; D(A))$, the sequence $P_n F(t, u_n)$ is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;H))$, and $P_n G(t, u_n)$ is bounded in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;H^{\otimes m}))$.

Thus with Alaoglu's theorem, we can ensure that there exists a subsequence $\{u_{n'}\} \subset \{u_n\}$, and seven elements $u \in L^p(\Omega, \mathcal{F}, P; L^2(0,T;D(A^{3/2}))) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;D(A)))$, $v \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;V)) \cap L^2(\Omega, \mathcal{F}, P; L^\infty(0,T;H))$, $B^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;V'))$, $F^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;H))$, $\rho_1 \in L^2(\Omega, \mathcal{F}_0, H)$, $\rho_2 \in L^2(\Omega, \mathcal{F}_0, D(A))$ and $G^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;H^{\otimes m}))$ such that:

$$u_{n'} \rightharpoonup u \quad \text{in } L^p\left(\Omega, \mathcal{F}, P; L^2\left(0,T;D\left(A^{3/2}\right)\right)\right) \cap L^p(\Omega, \mathcal{F}, P; L^\infty(0,T;D(A))), \tag{4.5}$$

$$v_{n'} \rightharpoonup v \quad \text{in } L^2\left(\Omega, \mathcal{F}, P; L^2(0,T;V)\right), \tag{4.6}$$

$$P_{n'}B(u_{n'}, v_{n'}) \rightharpoonup B^* \quad \text{in } L^2\left(\Omega, \mathcal{F}, P; L^2\left(0,T;V'\right)\right), \tag{4.7}$$

$$P_{n'}F(t, u_{n'}) \rightharpoonup F^* \quad \text{in } L^2\left(\Omega, \mathcal{F}, P; L^2(0,T;H)\right),$$
$$v_{n'}(0) \rightharpoonup \rho_1 \quad \text{in } L^2(\Omega, \mathcal{F}_0, H) \tag{4.8}$$
$$u_{n'}(0) \rightharpoonup \rho_2 \quad \text{in } L^2(\Omega, \mathcal{F}_0, D(A))$$

$$P_{n'}G(t, u_{n'}) \rightharpoonup G^* \quad \text{in } L^2\left(\Omega, \mathcal{F}, P; L^2\left(0,T;H^{\otimes m}\right)\right). \tag{4.9}$$

Using Remark 4.2 and the weak convergence above, we obtain from (3.1)

$$v(t) + \nu\int_0^t Av(s)\ ds + \int_0^t B^*(s)\ ds = v_0 + \int_0^t F^*(s)\ ds + \int_0^t G^*(s)dW(s) \tag{4.10}$$

for all $t \in [0,T]$, where $v(t) = u(t) + \alpha^2 Au(t)$ and $v_0 = u_0 + \alpha^2 Au_0$.

Referring then to results [21, 24, 25], we find that v has modification such that $v \in C(0,T;H)$ a.s. which implies that u has modification in $C(0,T;D(A))$ a.s. □

4.1.2. Proof of $B^* = B(u,v)$, $F^* = F(t,u)$ and $G^* = G(t,u)$

For simplicity we keep on denoting by $\{u_n\}$ the subsequence $\{u_{n'}\}$ in this step.

Let $(X(t))_{t\in[0,T]}$ be a process in the space $L^2(\Omega,\mathcal{F},P;L^2(0,T;V))$. Using the properties of A and of its eigenvectors $\{w_1,w_2,\ldots\}(\lambda_1,\lambda_2,\ldots$ are the corresponding eigenvalues), we have

$$\begin{aligned}
&\|P_nX(t)\| \le \|X(t)\|, \quad |P_nX(t)| \le |X(t)|, \quad |X(t) - P_nX(t)| \le |X(t)|,\\
\beta\|X(t) - P_nX(t)\|^2 &\le \langle AX(t) - AP_nX(t), X(t) - P_nX(t)\rangle_{V'}\\
&= \sum_{i=n}^{i=\infty}\lambda_i(X(t),w_i)^2\\
&\le \langle AX(t), X(t)\rangle_{V'}\\
&\le C\|X(t)\|^2.
\end{aligned} \tag{4.11}$$

Hence for $dP \times dt$ a.e. $(w,t) \in \Omega \times [0,T]$, we have

$$\lim_{n\to\infty}\|X(w,t) - P_nX(w,t)\|^2 = 0. \tag{4.12}$$

By the Lebesgue dominated convergence theorem, it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T \|X(t) - P_nX(t)\|^2 dt = 0,\\
&\lim_{n\to\infty} E\int_0^T \|X(t) - P_nX(t)\|^2 dt = 0,
\end{aligned} \tag{4.13}$$

$$\lim_{n\to\infty} E\|X(t) - P_nX(t)\|^2 = 0. \tag{4.14}$$

Applying this result to $X = v \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;V))$ or $X = u$, we have

$$P_n v \longrightarrow v \quad \text{in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;V)\Big), \tag{4.15}$$

$$P_n u \longrightarrow u \quad \text{in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;V)\Big). \tag{4.16}$$

With a candidate solution in hand, it remains to show that

$$B^* = B(u,v), \qquad F^* = F(t,u), \qquad G^* = G(t,u). \tag{4.17}$$

In the next lemma, we compare v and the sequence $v_n = u_n + \alpha^2 A u_n$, at least up to a stopping time $\tau_m \uparrow T$ a.s.; this is sufficient to deduce the existence result. Here, we are adapting techniques used in [10, 11].

Let $m \in \mathbf{N}^*$, consider the $\mathcal{F}_t$-stopping time τ_m defined by

$$\tau_m = \inf\left\{ t; |v(t)|^2 + \int_0^t \|v(s)\|^2 ds \geq m^2 \right\} \wedge T. \tag{4.18}$$

Notice that τ_m is increasing as a function of m and moreover $\tau_m \to T$ a.s.

Lemma 4.3. *One has*

$$\lim_{n\to\infty} E \int_0^{\tau_m} \|v_n(s) - v(s)\|^2 ds = 0. \tag{4.19}$$

Proof. Using (4.15), it suffices to prove that

$$\lim_{n\to\infty} E \int_0^{\tau_m} \|P_n v(s) - v_n(s)\|^2 \, ds = 0. \tag{4.20}$$

Using (3.1) and (4.10), the difference of $P_n v$ and v_n satisfies the relation

$$\begin{aligned} &d(P_n v - v_n) + [\nu A(P_n v - v_n) + P_n B^* - P_n B(u_n, v_n)]dt \\ &\quad = P_n(F^* - F(t,u_n))dt + P_n(G^* - G(t,u_n))dW. \end{aligned} \tag{4.21}$$

Let $\sigma(t) = \exp\{-n_1 t - n_2 \int_0^t \|v(s)\|^2 \, ds\}, 0 \leq t \leq T$, with n_1 and n_2 positive constants to be fixed later.

Applying Ito's formula to the process $\sigma(t)|P_n v - v_n|^2$, we have

$$\begin{aligned}
&\sigma(t)|P_n v(t) - v_n(t)|^2 + 2\beta\nu\int_0^t \sigma(t)\|P_n v(s) - v_n(s)\|^2 ds \\
&\quad\leq 2\int_0^t \sigma(s)\langle B^*(s) - B(u_n(s), v_n(s)), P_n v(s) - v_s\rangle_{V'} ds \\
&\qquad + 2\int_0^t \sigma(s)(F^*(s) - F(s, u_n(s)), P_n v(s) - v_n(s)) ds \\
&\qquad + 2\int_0^t \sigma(s)|P_n(G^*(s) - G(s, u_n(s)))|^2 ds \\
&2\int_0^t \sigma(s)(G^*(s) - G(s, u_n(s)), P_n v(s) - v_n(s)) dW - n_1\int_0^t \sigma(s)|P_n v(s) - v_n(s)|^2 ds \\
&\qquad - n_2\int_0^t \sigma(s)\|v(s)\|^2|P_n v(s) - v_n(s)|^2 ds.
\end{aligned} \tag{4.22}$$

We are going to estimate the first three terms of the right-hand side of (4.22).

For the first term, using the cancellation property (2.4) and (2.8), we have

$$\begin{aligned}
&\langle B^* - B(u_n, v_n), P_n v - v_n\rangle_{V'} \\
&\quad = \langle B^*, P_n v - v_n\rangle_{V'} + \langle B(u_n - P_n u, P_n v), v_n - P_n v\rangle_{V'} + \langle B(P_n u, P_n v), v_n - P_n v\rangle_{V'} \\
&\quad \leq \langle B^*, P_n \mathrm{v} - v_n\rangle_{V'} + C|u_n - P_n u|^{1/4}\|u_n - P_n u\|^{3/4}|P_n v|^{1/4}\|P_n v\|^{3/4}\|v_n - P_n v\| \\
&\qquad + \langle B(P_n u, P_n v), v_n - P_n v\rangle_{V'} \\
&\quad \leq \langle B^*, P_n v - v_n\rangle_{V'} + \frac{C}{2\beta}\|v\|^2|v_n - P_n v|^2 + \frac{\beta}{2}\|v_n - P_n v\|^2 + \langle B(P_n u, P_n v), v_n - P_n v\rangle_{V'}.
\end{aligned} \tag{4.23}$$

For the term involving F^* and F, using the Lipschitz conditions on F, we have

$$\begin{aligned}
2(F^* - F(t, u_n), P_n v - v_n) &\leq 2(F^* - F(t, u), P_n v - v_n) + 2(F(t, u) - F(t, P_n u), P_n v - v_n) \\
&\qquad + 2L_F|P_n u - u_n||P_n v - v_n| \\
&\leq 2(F^* - F(t, u), P_n v - v_n) + 2(F(t, u) - F(t, P_n u), P_n v - v_n) \\
&\qquad + 2CL_F|P_n v - v_n|^2.
\end{aligned} \tag{4.24}$$

For the term involving G^* and G, using the Lipschitz conditions on G, we have

$$\begin{aligned}|P_n(G^*-G(t,u_n))|^2 &\leq 2L_G^2|P_nu-u_n|^2+2L_G^2|u-P_nu|^2+2(G^*-G(t,u),P_n(G^*-G(t,u_n)))\\&\quad-|P_n(G^*-G(t,u))|^2\\&\leq 2L_G^2|P_nv-v_n|^2+2L_G^2|u-P_nu|^2+2(G^*-G(t,u),P_n(G^*-G(t,u_n)))\\&\quad-|P_n(G^*-G(t,u))|^2.\end{aligned}\tag{4.25}$$

Taking into account (4.23)–(4.25), we obtain from (4.22)

$$\begin{aligned}&\sigma(t)|P_nv(t)-v_n(t)|^2+2\beta\int_0^t\sigma(s)\|P_nv(s)-v_n(s)\|^2ds+2\int_0^t\sigma(s)|P_n(G^*(s)-G(s,u(s)))|^2ds\\&\leq 2\int_0^t\sigma(s)\langle B^*(s),P_nv(s)-v_n(s)\rangle_{V'}\,ds+\frac{C}{\beta}\int_0^t\sigma(s)\|v(s)\|^2|v_n(s)-P_nv(s)|^2ds\\&\quad+\beta\int_0^t\sigma(s)\|P_nv(s)-v_n(s)\|^2ds\\&\quad+2\int_0^t\sigma(s)\langle B(P_nu(s),P_nv(s)),v_n(s)-P_nv(s)\rangle_{V'}\,ds+4CL_F\int_0^t\sigma(s)|P_nv(s)-v_n(s)|^2ds\\&\quad+4\int_0^t\sigma(s)(F^*(s)-F(s,u(s)),P_nv(s)-v_n(s))ds\\&\quad+4\int_0^t\sigma(s)(F(s,u(s))-F(s,P_nu(s)),P_nv(s)-v_n(s))ds\\&\quad+4L_G^2\int_0^t\sigma(s)|P_nv(s)-v_n(s)|^2ds+4L_G^2\int_0^t\sigma(s)|\mathrm{u}(s)-P_nu(s)|^2ds\\&\quad+4\int_0^t\sigma(s)(G^*(s)-G(s,u(s)),P_n(G^*(s)-G(s,u(s))))ds\\&\quad-n_1\int_0^t\sigma(s)|P_nv(s)-v_n(s)|^2ds-n_2\int_0^t\sigma(s)\|v(s)\|^2|P_nv(s)-v_n(s)|^2\\&\quad+2\int_0^t\sigma(s)(G^*(s)-G(s,u_n(s)),P_nv(s)-v_{\mathrm{n}}(s))dW.\end{aligned}\tag{4.26}$$

Therefore, if we take $n_1 = 4CL_F + 4L_G^2$ and $n_2 = C/\beta\nu$, we obtain from (4.26)

$$\begin{aligned}
&E\sigma(\tau_m)|P_n v(\tau_m) - v_n(\tau_m)|^2 + \frac{3\beta\nu}{2}E\int_0^{\tau_m}\sigma(s)\|P_n v(s) - v_n(s)\|^2 ds \\
&\quad + 2E\int_0^{\tau_m}\sigma(s)|P_n(G^*(s) - G(s,u(s)))|^2 ds \\
&\leq 2E\int_0^{\tau_m}\sigma(s)\langle B^*(s), P_n v(s) - v_n(s)\rangle_{V'} ds \\
&\quad + 2E\int_0^{\tau_m}\sigma(s)\langle B(P_n u(s), P_n v(s)), v_n(s) - P_n v(s)\rangle_{V'} ds \\
&\quad + 4E\int_0^{\tau_m}\sigma(s)(F^*(s) - F(s,u(s)), P_n v(s) - v_n(s)) ds \\
&\quad + 4E\int_0^{\tau_m}\sigma(s)(F(s,u(s)) - F(s,P_n u(s)), P_n v(s) - v_n(s)) ds \\
&\quad + 4L_G^2 E\int_0^{\tau_m}\sigma(s)|u(s) - P_n u(s)|^2 ds \\
&\quad + 4E\int_0^{\tau_m}\sigma(s)(G^*(s) - G(s,u(s)), P_n(G^*(s) - G(s,u(s)))) ds.
\end{aligned} \tag{4.27}$$

Next, we are going to prove the convergence to 0 of each term on the right-hand side of (4.27). Here we use some basic convergence principles from functional analysis [12, 13].

For the first two terms, we have

$$\begin{aligned}
&E\int_0^{\tau_m}\sigma(s)\langle B(P_n u(s), P_n v(s)) - B^*(s), v_n(s) - P_n v(s)\rangle_{V'} ds \\
&\quad = E\int_0^{\tau_m}\sigma(s)\langle B(P_n u(s), P_n v(s)) - B(u(s), v(s)), v_n(s) - P_n v(s)\rangle_{V'} ds \\
&\qquad + E\int_0^{\tau_m}\sigma(s)\langle B(u(s), v(s)) - B^*(s), v_n(s) - P_n v(s)\rangle_{V'} ds.
\end{aligned} \tag{4.28}$$

From the properties of B, we have

$$\begin{aligned}
\|B(P_n u, P_n v) - B(u,v)\|_{V'} &\leq \|B(P_n u - u, P_n v)\|_{V'} + \|B(u, P_n v - v)\|_{V'} \\
&\leq (\|P_n u - u\|\|P_n v\| + \|u\|\|P_n v - v\|).
\end{aligned} \tag{4.29}$$

We have from (4.15) and (4.16)

$$\begin{gathered}
\left\|I_{[0,\tau_m]}\sigma(t)B(P_n u, P_n v) - B(u,v)\right\|_{V'} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,\ dt \times dP - a.e., \\
\left\|I_{[0,\tau_m]}\sigma(t)(B(P_n u, P_n v) - B(u,v))\right\|_{V'} \leq C\|u(t)\|\|v(t)\| \in L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;\mathbb{R})\Big).
\end{gathered} \tag{4.30}$$

Using (4.6) and (4.15), we have

$$v_n - P_n v \rightharpoonup 0 \quad \text{in } L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;V)\Big). \tag{4.31}$$

Applying the results of weak convergence (see [12, 13]), it follows from (4.30) and (4.31) that

$$\lim_{n\to\infty} E\int_0^{\tau_m} \sigma(s)\langle B(P_n u, P_n v) - B(u,v), v_n(s) - P_n v(s)\rangle_{V'} ds = 0. \tag{4.32}$$

Also as $I_{[0,\tau_m]}\sigma(t)B(u,v) - B^* \in L^2(\Omega, \mathcal{F}, P; L^2(0,T;V'))$, we have from (4.31)

$$\lim_{n\to\infty} E\int_0^{\tau_m} \sigma(s)\langle B(u(s), v(s)) - B^*(s), v_n(s) - P_n v(s)\rangle_{V'} ds = 0. \tag{4.33}$$

On the other hand, from (4.16), the Lipschitz conditions on F, G and the fact that $v_n - P_n v \rightharpoonup 0$ in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;H))$, we have

$$\begin{aligned}
&\lim_{n\to\infty} E\int_0^{\tau_m} \sigma(s)(G(s,u(s)) - G(s,P_n u(s)), v_n(s) - P_n v(s)) ds = 0,\\
&\lim_{n\to\infty} E\int_0^{\tau_m} \sigma(s)(F(s,u(s)) - F(s,P_n u(s)), v_n(s) - P_n v(s)) ds = 0.
\end{aligned} \tag{4.34}$$

Again from (4.31) and the fact that

$$\begin{aligned}
&F^* - F(t,u) \in L^2\Big(\Omega, \mathcal{F}, P; L^2(0,T;H)\Big),\\
&G^* - G(t,u) \in L^2\Big(\Omega, \mathcal{F}, P; L^2\big(0,T;H^{\otimes m}\big)\Big),
\end{aligned} \tag{4.35}$$

we have

$$\begin{aligned}
&\lim_{n\to\infty} E\int_0^{\tau_m} \sigma(s)(F^*(s) - F(s,u(s)), v_n(s) - P_n v(s)) ds = 0,\\
&\lim_{n\to\infty} E\int_0^{\tau_m} \sigma(s)(G^*(s) - G(s,u(s)), v_n(s) - P_n v(s)) ds = 0.
\end{aligned} \tag{4.36}$$

As

$$P_n(G^* - G(t,u_n)) \rightharpoonup 0 \quad \text{in} \quad L^2\Big(\Omega, \mathcal{F}, P; L^2\big(0,T;H^{\otimes m}\big)\Big), \tag{4.37}$$

we also have

$$\lim_{n\to\infty} E\int_0^{\tau_m} \sigma(s)(G^*(s) - G(s,u(s)), P_n(G^*(s) - G(s,u_n(s)))) ds = 0. \tag{4.38}$$

From (4.32)–(4.38), and the fact that

$$\exp(-n_1 T - n_2 m) \leq I_{[0,\tau_m]\sigma(t)} \leq 1, \tag{4.39}$$

we obtain from (4.27)

$$\lim_{n\to\infty} E\left(|P_n v(\tau_m) - v_n(\tau_m)|^2\right) = 0, \tag{4.40}$$

$$\lim_{n\to\infty} E\int_0^{\tau_m} \|P_n v(s) - v_n(s)\|^2 ds = 0, \tag{4.41}$$

$$E\int_0^{\tau_m} |G^*(s) - G(s, u(s))|^2 ds = 0. \tag{4.42}$$

Now from (4.42) and the fact that the sequence τ_m tends to T, we have

$$G^*(t) = G(t, u(t)) \tag{4.43}$$

as elements of the space $L^2(\Omega, \mathcal{F}, P; L^2(0, T; H^{\otimes m}))$.

Also observe that (4.40) and (4.15) imply that

$$v_n I_{[0,\tau_m]} \longrightarrow v I_{[0,\tau_m]} \quad \text{in } L^2\left(\Omega, \mathcal{F}, P; L^2(0, T; V)\right), \tag{4.44}$$

where $I_{[0,\tau_m]}$ is the indicator function of $[0, \tau_m]$. Let $w \in V$. We have the following estimate from B:

$$\begin{aligned} &|\langle B(u,v) - P_n B(u_n, v_n), w\rangle_{V'}| \\ &\quad\leq |\langle B(u,v) - B(u_n, v_n), w\rangle_{V'}| + |\langle (I - P_n) B(u_n, v_n), w\rangle_{V'}| \\ &\quad\leq C(\|u - u_n\|\|v\| + \|v_n - v\|\|v_n\|)\|w\| + C\|(I - P_n)w\|\|u_n\|\|v_n\|. \end{aligned} \tag{4.45}$$

Thus from (4.45) and using Hölder's inequality, we have

$$\begin{aligned} &E\int_0^{\tau_m} \langle B(u(s), v(s)) - P_n B(u_n(s), v_n(s)), w\rangle_{V'} ds \\ &\quad\leq C\left(E\int_0^{\tau_m} \|u(s) - u_n(s)\|^2 ds\right)^{1/2} \left(E\int_0^T \|v(s)\|^2 ds\right)^{1/2} \\ &\qquad + \left(E\int_0^{\tau_m} \|v_n(s) - v(s)\|^2 ds\right)^{1/2} \left(E\int_0^T \|v_n(s)\|^2 ds\right)^{1/2} \\ &\qquad + C\|(I - P_n)w\|\left(E\int_0^T \|u_n(s)\|^2 ds\right)^{1/2} \left(E\int_0^T \|v_n(s)\|^2 ds\right)^{1/2}. \end{aligned} \tag{4.46}$$

Consequently, by (4.44) and (4.46), we have

$$\lim_{n\to\infty} E\int_0^{\tau_m} \langle B(u(s),v(s)) - P_n B(u_n(s),v_n(s)), w\rangle_{V'} ds = 0. \tag{4.47}$$

Taking into account (4.7), it follows from (4.47) that

$$E\int_0^{\tau_m} \langle B(u(s),v(s)) - B^*(s), z(s)\rangle_{V'} ds = 0 \tag{4.48}$$

for all $z \in \mathfrak{D}_V(\Omega \times [0,T])$, where $\mathfrak{D}_V(\Omega \times [0,T])$ is a set of $\psi \in L^\infty(\Omega, \mathcal{F}, P; L^\infty(0,T;V))$ with

$$\psi = w\phi, \quad \phi \in L^\infty(\Omega \times [0,T];\mathbb{R}), \quad w \in V. \tag{4.49}$$

Therefore, as τ_m tends to T and $\mathfrak{D}_V(\Omega \times [0,T])$ is dense in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;V))$, we obtain from (4.48) that $B(u(t),v(t)) = B^*(t)$ as elements of the space $L^2(\Omega, \mathcal{F}, P; L^2(0,T;V'))$.

Analogously, using the Lipschitz condition on F and (4.44), we have $F(t,u(t)) = F^*(t)$ as elements of the space $L^2(\Omega, \mathcal{F}, P; L^2(0,T;H))$.

And the existence result follows. □

4.2. Uniqueness

Let u_1 and u_2 be two solutions of problem (2.15), which have in $D(A)$ almost surely continuous trajectories with the same initial data u_0. Denote

$$\begin{gathered} v_1 = u_1 + \alpha^2 A u_1, \qquad v_2 = u_2 + \alpha^2 A u_2, \\ v = v_1 - v_2, \quad u = u_1 - u_2. \end{gathered} \tag{4.50}$$

By Ito's formula, we have

$$\begin{aligned} &|v(t)|^2 + 2\int_0^t \langle Av(s), v(s)\rangle_{V'} + 2\int_0^t \langle B(u_1(s),v_1(s)) - B(u_2(s),v_2(s)), v(s)\rangle_{V'} \\ &\quad = 2\int_0^t (F(s,u_1(s)) - F(s,u_2(s)), v(s))ds + 2\int_0^t (G(s,u_1(s)) - G(s,u_2(s)), v(s))ds \\ &\qquad + \int_0^t |G(s,u_1(s)) - G(s,u_2(s))|^2_{H^{\otimes m}} ds. \end{aligned} \tag{4.51}$$

Take $\lambda > 0$ to be fixed later and define

$$\sigma(t) = \exp\left\{ -\frac{b}{\beta}\int_0^t \|v_1(s)\|^2 ds - \lambda t \right\}. \tag{4.52}$$

Applying Ito's formula to the real-valued process $\sigma(t)|v(t)|^2$, we obtain from(4.51)

$$\begin{aligned}
&\sigma(t)|v(t)|^2 + 2\beta\nu\int_0^t \sigma(s)\|v(s)\|^2 ds \\
&\quad\leq 2\int_0^t \sigma(s)\langle B(u(s), v_1(s)), v(s)\rangle_{V'}\ ds + 2\int_0^t \sigma(s)(F(s,u_1(s)) - F(s,u_2(s)), v(s))ds \\
&\quad\quad + 2\int_0^t \sigma(s)(G(s,u_1(s)) - G(s,u_2(s)), v(s))dW(s) \\
&\quad\quad + \int_0^t \sigma(s)|G(s,u_1(s)) - G(s,u_2(s))|^2_{H^{\otimes m}} ds \\
&\quad\quad - \int_0^t \frac{b}{\beta}\|v_1(s)\|^2|v(s)|^2\sigma(s)ds - \int_0^t \lambda\sigma(s)|v(s)|^2 ds.
\end{aligned} \tag{4.53}$$

But from (2.8), we have

$$\begin{aligned}
&\langle B(u(s), v_1(s)), v(s)\rangle_{V'} \\
&\quad\leq C|u(s)|^{1/4}\|u(s)\|^{3/4}\|v_1(s)\|^{3/4}\|v(s)\| \\
&\quad\leq C|v(s)|^{1/4}|v(s)|^{3/4}\|v_1(s)\|\|v(s)\| \\
&\quad\leq \frac{C}{2\nu\beta}\|v_1(s)\|^2|v(s)|^2 + \frac{\beta\nu}{2}\|v(s)\|^2, \\
&(F(s,u_1(s)) - F(s,u_2(s)), v(s)) \leq L_F|v(s)|^2, \\
&|G(s,u_1(s)) - G(s,u_2(s))|_{H^{\otimes m}} \leq L_G|v(s)|.
\end{aligned} \tag{4.54}$$

We then obtain from(4.53)

$$\begin{aligned}
&\sigma(t)|v(t)|^2 + 2\beta\nu\int_0^t \sigma(s)\|v(s)\|^2 ds \\
&\quad\leq \frac{C}{\beta}\int_0^t \sigma(s)\|v_1(s)\|^2|v(s)|^2\ ds + \frac{\nu\beta}{2}\int_0^t \sigma(s)\|v(s)\|^2 ds + 2L_F\int_0^t \sigma(s)|v(s)|^2 ds \\
&\quad\quad + 2\int_0^t \sigma(s)(G(s,u_1(s)) - G(s,u_2(s)), v(s))\ dW(s) + L_G^2\int_0^t \sigma(s)|v(s)|^2 ds \\
&\quad\quad - \int_0^t \frac{b}{\beta}\|v_1(s)\|^2|v(s)|^2\sigma(s)\ ds - \int_0^t \lambda\sigma(s)|v(s)|^2 ds.
\end{aligned} \tag{4.55}$$

Taking $\lambda = L_G^2$ and $b = C$, we obtain from(4.55)

$$\begin{aligned} &\sigma(t)|v(t)|^2 + \frac{3\nu\beta}{2}\int_0^t \sigma(s)\|v(s)\|^2 ds \\ &\quad \le 2L_F\int_0^t \sigma(s)|v(s)|^2 ds + 2\int_0^t \sigma(s)(G(s,u_1(s)) - G(s,u_2(s)), v(s))dW(s) \end{aligned} \tag{4.56}$$

for all $t \in [0,T]$.

As $0 < \sigma(t) \le 1$, the expectation of the stochastic integral in (4.56) vanishes, and

$$E\sigma(t)|v(t)|^2 \le 2L_G E\int_0^t \sigma(s)|v(s)|^2 ds, \tag{4.57}$$

for all $t \in [0,T]$. The Gronwall's lemma implies that

$$|v(t)| = 0, \qquad P - a.s. \quad \forall t \in [0,T], \tag{4.58}$$

in particular

$$u(t) = 0, \qquad P - a.s. \quad \forall t \in [0,T]. \tag{4.59}$$

This complete the proof of the uniqueness.

5. Proof of Theorem 2.4

To prove the convergence result of Theorem 2.4, we need the following lemma which is proved in [10, 11].

Lemma 5.1. *Let $\{Q_n, n \ge 1\}$ be a sequence of continuous real-valued processes in $L^2(\Omega, \mathcal{F}, P; L^2(0,T;\mathbb{R}))$, and let $\{\sigma_m; m \ge 1\}$ be a sequence of $\mathcal{F}_t$-stopping times such that σ_m is increasing to T, $\sup_{n\ge1} E|Q_n(T)|^2 < \infty$, and $\lim_{n\to\infty} E|Q_n(\sigma_m)| = 0$ for all $n \ge 1$. Then $\lim_{n\to\infty} E|Q_n(T)| = 0$.*

It follows from (4.41) and (4.15) that

$$\lim_{n\to\infty} E\int_0^{\tau_m} \|v_n(t) - v(t)\|^2 dt = 0. \tag{5.1}$$

Also from (4.40) and (4.14), we have

$$\lim_{n\to\infty} E|v_n(\tau_m) - v(\tau_m)|^2 = 0. \tag{5.2}$$

Applying the preceding lemma to $Q_n(t) = \int_0^t \|v_n(s) - v(s)\|^2 ds$ and $\sigma_m = \tau_m$, and taking into account the estimate of v_n in Lemma 3.3, (5.1), and the uniqueness of v (or u), one obtains that the whole sequence v_n defined by (3.1) satisfies

$$\lim_{n\to\infty} E\int_0^t \|v_n(s) - v(s)\|^2 ds = 0 \tag{5.3}$$

for all $t \in [0,T]$. Next, using the expression of v_n and v, we deduce that

$$\lim_{n\to\infty} E\int_0^t \|u_n(s) - u(s)\|^2_{D(A^{3/2})} ds = 0. \tag{5.4}$$

Analogously, applying the lemma to $Q_n(t) = |v_n(t) - v(t)|^2$ and $\sigma_m = \tau_m$, and taking into account the estimate of v_n in Lemma 3.2, (5.2), and the uniqueness of u, we have that the whole sequence v_n defined by (3.1) satisfies $\lim_{n\to\infty} E|v_n(t) - v(t)|^2 = 0$. Using the expression of v_n and v, we have $\lim_{n\to\infty} E\|u_n(t) - u(t)\|^2_{D(A)} = 0$ for all $t \in [0,T]$. This complete the proof of Theorem 2.4.

6. Asymptotic Behavior of Strong Solutions for the 3D Stochastic Leray-α as α Approaches Zero

The purpose of this section is to study the behavior of strong solutions for the 3D stochastic Leray-α model as α goes to zero. Therefore, we study the weak compactness of strong solutions of the 3D stochastic Leray-α equations as α approaches zero. One of the crucial point is to show that

$$E \sup_{0\le|\theta|\le\delta\le 1} \int_\delta^{T-\delta} |u_\alpha(t+\theta) - u_\alpha(t)|^2_{D(A)'}\, dt \le C\delta, \tag{6.1}$$

where C is a constant independent of α. To do this, we adopt the method developed for the deterministic 3D Leray-α equations [2]. In this method, an important role is played by the operator $(I + \alpha^2 A)^{-1}$. Here our line of investigation is inspired by [5, 6, 9].

6.1. Tightness of Strong Solutions for the 3D Stochastic Leray-α Equations

In this subsection, we prove the tightness of strong solutions of the 3D stochastic Leray-α equations as α approaches zero. The main result of this subsection is the following lemma.

Lemma 6.1. *Suppose that hypotheses* (2.13) *hold, and* $u_0 \in L^2(\Omega, \mathcal{F}_0, P; D(A))$. *Let* u_α *be a strong solution for the* 3*D stochastic Leray-*α *equations. One has*

$$E \sup_{0\le|\theta|\le\delta\le 1} \int_\delta^{T-\delta} |u_\alpha(t+\theta) - u_\alpha(t)|^2_{D(A)'} dt \le C\delta, \tag{6.2}$$

where C *is a constant independent of* α.

Proof. We recall that $D(A)' = D(A^{-1})$.

From (2.15), we have

$$d\left(I+\alpha^2 A\right)u_\alpha + \nu A\left(u_\alpha + \alpha^2 A u_\alpha\right)dt + B\left(u_\alpha, u_\alpha + \alpha^2 A u_\alpha\right)\ dt = F(t,u_\alpha)\ dt + G(t,u_\alpha)dW. \tag{6.3}$$

We recall that $I+\alpha^2 A$ is an isomorphism from $D(A)$ to H and

$$\left\|\left(I+\alpha^2 A\right)^{-1}\right\|_{\mathcal{L}(H,H)} \le 1. \tag{6.4}$$

From (6.3), we have

$$du_\alpha + \nu A u_\alpha dt + \left(I+\alpha^2 A\right)^{-1} B(u_\alpha, v_\alpha)dt = \left(I+\alpha^2 A\right)^{-1} F(t,u_\alpha)dt + \left(I+\alpha^2 A\right)^{-1} G(t,u_\alpha)dW, \tag{6.5}$$

where $v_\alpha = u_\alpha + \alpha^2 A u_\alpha$.

We deduce that

$$\begin{aligned}
&\left|A^{-1}(u_\alpha(t+\theta) - u_\alpha(t))\right| \\
&\int_t^{t+\theta}\left(\left|A^{-1}\left(I+\alpha^2 A\right)^{-1}F(\tau,u_\alpha(\tau))\right| + \nu|u_\alpha(\tau)| + \left|A^{-1}\left(I+\alpha^2 A\right)^{-1}B(u_\alpha(\tau),v_\alpha(\tau))\right|\right)d\tau \\
&\quad + \left|\int_t^{t+\theta} A^{-1}\left(I+\alpha^2 A\right)^{-1}G(\tau,u_\alpha(\tau))dW(\tau)\right|.
\end{aligned} \tag{6.6}$$

We estimate the first terms of the left-hand side of (6.6) using (2.7) and the Lipschitz condition on F

$$\begin{aligned}
\left|A^{-1}\left(I+\alpha^2 A\right)^{-1}B(u_\alpha(\tau),v_\alpha(\tau))\right| &\le \left|A^{-1}B(u_\alpha(\tau),v_\alpha(\tau))\right| \le C|u_\alpha(\tau)|\|v_\alpha(\tau)\|, \\
\left|A^{-1}\left(I+\alpha^2 A\right)^{-1}F(\tau,u_\alpha(\tau))\right| &\le \left|A^{-1}F(\tau,u_\alpha(\tau))\right| \le C(1+|u_\alpha(\tau)|).
\end{aligned} \tag{6.7}$$

Collecting these previous inequalities and taking the square in (6.6), we have

$$\begin{aligned}\left|A^{-1}(u_\alpha(t+\theta)-u_\alpha(t))\right|^2 \le C\theta^2 &+ C_1\left(\int_t^{t+\theta}|u_\alpha(\tau)|d\tau\right)^2 + \nu^2\left(\int_t^{t+\theta}|u_\alpha(\tau)|d\tau\right)^2\\ &+ C\left(\int_t^{t+\theta}|u_\alpha(\tau)|\|v_\alpha(\tau)\|d\tau\right)^2\\ &+ \left|\int_t^{t+\theta}A^{-1}\left(I+\alpha^2A\right)^{-1}G(\tau,u_\alpha(\tau))dW(\tau)\right|^2.\end{aligned} \tag{6.8}$$

For fixed δ, taking the supremun over $\theta \le \delta$ yields

$$\begin{aligned}&\sup_{0\le\theta\le\delta}\left|A^{-1}(u_\alpha(t+\theta)-u_\alpha(t))\right|^2\\ &\quad\le C\delta^2 + TC_1\delta^2\sup_{\tau\in[0,T]}|u_\alpha(\tau)|^2 + C_4\sup_{\tau\in[0,T]}|u_\alpha(\tau)|^2\left(\int_t^{t+\delta}\|v_\alpha(\tau)\|d\tau\right)^2\\ &\qquad+\sup_{0\le\theta\le\delta}\left|\int_t^{t+\theta}A^{-1}\left(I+\alpha^2A\right)^{-1}G(\tau,u_\alpha(\tau))dW(\tau)\right|^2.\end{aligned} \tag{6.9}$$

For t, we integrate between δ and $T-\delta$ and take the expectation. We deduce

$$\begin{aligned}&E\sup_{0\le\theta\le\delta}\int_\delta^{T-\delta}\left|A^{-1}(u_\alpha(t+\theta)-u_\alpha(t))\right|^2dt\\ &\quad\le C\delta^2 + TC\delta^2E\sup_{\tau\in[0,T]}|u_\alpha(\tau)|^2\\ &\qquad+C_4E\sup_{\tau\in[0,T]}|u_\alpha(\tau)|^2\int_\delta^{T-\delta}\left(\int_t^{t+\delta}\|v_\alpha(\tau)\|d\tau\right)^2dt\\ &\qquad+E\int_\delta^{T-\delta}\sup_{0\le\theta\le\delta}\left|\int_t^{t+\theta}A^{-1}\left(I+\alpha^2A\right)^{-1}G(\tau,u_\alpha(\tau))dW(\tau)\right|^2dt.\end{aligned} \tag{6.10}$$

By Hölder's inequality, we have

$$
\begin{aligned}
&E \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^2 \int_\delta^{T-\delta} \left(\int_t^{t+\delta} \|v_\alpha(\tau)\| d\tau \right)^2 dt \\
&\leq \delta^2 E \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^2 \int_\delta^{T-\delta} \|v_\alpha(\tau)\|^2 d\tau \\
&\leq \delta^2 \left(E \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^4 \right)^{1/2} \left[E \left(\int_0^T \|v_\alpha(\tau)\|^2 d\tau \right)^2 \right]^{1/2}.
\end{aligned}
\tag{6.11}
$$

Using the estimates of Lemmas 3.1, 3.2, 3.3, we obtain

$$
E \sup_{\tau\in[0,T]} |u_\alpha(\tau)|^2 \int_\delta^{T-\delta} \left(\int_t^{t+\delta} \|v_\alpha(\tau)\| d\tau \right)^2 dt \leq C\delta^2,
\tag{6.12}
$$

where C is a constant independent of α.

Next, using Martingale's inequality, we have

$$
\begin{aligned}
&E \int_\delta^{T-\delta} \sup_{0\leq\theta\leq\delta} \left| \int_t^{t+\theta} A^{-1} \left(I + \alpha^2 A \right)^{-1} G(s, u_\alpha(s)) dW(s) \right|^2 dt \\
&\leq E \int_\delta^{T-\delta} \left(\int_t^{t+\delta} \left| A^{-1} \left(I + \alpha^2 A \right)^{-1} G(s, u_\alpha(s)) \right|^2 ds \right) dt \\
&\leq CE \int_0^T \left(\int_t^{t+\delta} \left(1 + |u_\alpha(s)|^2 \right) ds \right) dt \\
&\leq C\delta.
\end{aligned}
\tag{6.13}
$$

Collecting these results, we finally obtain

$$
E \sup_{0\leq\theta\leq\delta\leq 1} \int_\delta^{T-\delta} |u_\alpha(t+\theta) - u_\alpha(t)|^2_{D(A)'} dt \leq C\delta,
\tag{6.14}
$$

where C is a constant independent of α. □

Remark 6.2. From Lemma 3.2, we have

$$
E \sup_{t\in[0,T]} |u_\alpha(\mathrm{t})|^p \leq C_p.
\tag{6.15}
$$

Also from Lemma 3.1, we have

$$E\int_0^T \|u_\alpha(s)\|^2 ds \leq C, \tag{6.16}$$

where C is constant independent of α.

From the estimate of Lemma 6.1 and Remark 6.2, we derive the following lemma which will be useful to prove the tightness of u_α.

Lemma 6.3. *Let ν_n and μ_n be two sequences of positives real number which tend to* 0 *as $n \to \infty$. The injection of*

$$\mathfrak{D} = \left\{ q \in L^\infty(0,T;H) \cap L^2(0,T;V); \sup_n \frac{1}{\nu_n} \sup_{|\theta|\leq\mu_n} \left(\int_{\mu_n}^{T-\mu_n} \left| q(t+\theta) - q(t) \right|^2_{D(A)'} dt \right)^{1/2} < \infty \right\} \tag{6.17}$$

in $L^2(0,T;H)$ is compact.

Proof. Its proof is carried out by the methods used in [5, 6, 9]. □

We define

$$S = C(0,T;R^m) \times L^2(0,T;H) \tag{6.18}$$

equipped with the Borel σ-algebra $B(S)$.

For $\alpha \in (0,1)$, let

$$\Phi : \Omega \longrightarrow S : \omega \longmapsto (W(\omega,\cdot), u_\alpha(\omega,\cdot)). \tag{6.19}$$

For each $\alpha \in (0,1)$, we introduce a probability measure Π_α on $(S, B(S))$ by

$$\Pi_\alpha(A) = P\left(\Phi^{-1}(A)\right), \tag{6.20}$$

where $A \in B(S)$.

In the next proposition, using the preceding lemma, we can prove the tightness of Π_α. Its proof is carried out by the methods in [26].

Proposition 6.4. *The family of probability measures $\{\Pi_\alpha; \alpha \in (0,1)\}$ is tight in S.*

6.2. Approximation of the Stochastic 3D Navier-Stokes Equations

In this section, we prove that the weak solutions of the stochastic 3D Navier-Stokes equations is obtained by a sequence of solutions of the 3D stochastic Leray-α model as α approaches zero. The result also gives us a new construction of the weak solutions for the 3D stochastic Navier-Stokes equations.

6.2.1. Application of Prokhorov's and Skorokhod's Results

From the tightness property of $\{\Pi_\alpha; 0 < \alpha \leq 1\}$ and Prokhorov's theorem (see [27]), we have that there exists a subsequence $\{\Pi_{\alpha_j}\}$ and a measure Π such that $\Pi_{\alpha_j} \to \Pi$ weakly. By Skorokhod's theorem (see [28]), there exist a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ and random variables $(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j})$, $(\widetilde{W}, \tilde{u})$ on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$ with values in S such that:

$$\begin{gathered}\text{the law of } \left(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j}\right) \text{ is } \Pi_{\alpha_j}, \\ \text{the law of } \left(\widetilde{W}, \tilde{u}\right) \text{ is } \Pi, \\ \left(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j}\right) \longrightarrow \left(\widetilde{W}, \tilde{u}\right) \text{ in } S \ \overline{P} - a.s.\end{gathered} \tag{6.21}$$

Hence $\{\widetilde{W}_{\alpha_j}\}$ is a sequence of a m-dimensional standard Wiener process.

Let

$$\overline{\mathcal{F}}_t = \sigma\left\{\widetilde{W}(s), \tilde{u}(s) : s \leq t\right\}. \tag{6.22}$$

Arguing as in [5, 9], we can prove that $\widetilde{W}$ is a m-dimensional $\overline{\mathcal{F}}_t$ standard Wiener process and the pair $(\widetilde{W}_{\alpha_j}, \tilde{u}_{\alpha_j})$ satisfies

$$\begin{aligned}&\left(\tilde{v}_{\alpha_j}(t), \Phi\right) + \nu \int_0^t \left(\tilde{v}_{\alpha_j}(s), A\Phi\right) ds + \int_0^t B\left(\tilde{u}_{\alpha_j}(s), \tilde{v}_{\alpha_j}(s), \Phi\right) ds \\ &\quad = \left(u_0 + \alpha_j^2 A u_0, \Phi\right) + \int_0^t \left(F\left(s, \tilde{u}_{\alpha_j}(s)\right), \Phi\right) ds + \left(\int_0^t G\left(s, \tilde{u}_{\alpha_j}(s)\right) d\widetilde{W}_{\alpha_j}(s), \Phi\right),\end{aligned} \tag{6.23}$$

for all $\Phi \in \mathcal{V}$, where

$$\tilde{v}_{\alpha_j}(s) = \tilde{u}_{\alpha_j}(s) + \alpha_j^2 A \tilde{u}_{\alpha_j}(s). \tag{6.24}$$

The main result of this section is the following theorem.

Theorem 6.5. *Suppose that hypotheses* (2.13) *hold, and* $u_0 \in D(A)$. *Then there is a subsequence of* $\tilde{u}_{\alpha_j}$ *denoted by the same symbol such that as* $\alpha_j \to 0$, *one has*

$$\begin{gathered}\tilde{u}_{\alpha_j} \longrightarrow \tilde{u} \quad \text{strongly in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;H)\right), \\ \tilde{u}_{\alpha_j} \longrightarrow \tilde{u} \quad \text{weakly in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;V)\right), \\ \tilde{v}_{\alpha_j} \longrightarrow \tilde{u} \quad \text{strongly in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;H)\right),\end{gathered} \tag{6.25}$$

where $(\overline{\Omega}, \overline{\mathcal{F}}, (\overline{\mathcal{F}}_t)_{t\in[0,T]}, \overline{P}, \widetilde{W}, \tilde{u})$ *is a weak solution for the 3D stochastic Navier-Stokes equations with the initial value* $u(0) = u_0$. *(See [5] for the definition of weak solution of the 3D stochastic Navier-Stokes equations).*

Proof. From (6.23), it follows that $\tilde{u}_{\alpha_j}$ satisfies the estimates

$$\widetilde{E}\sup_{0\le s\le T}\left|\tilde{u}_{\alpha_j}(s)\right|^p \le C_p; \tag{6.26}$$

$$\begin{aligned}
&\widetilde{E}\sup_{0\le s\le T}\left|\tilde{v}_{\alpha_j}(s)\right|^p \le C_p, \quad \widetilde{E}\sup_{0\le\theta\le\delta}\int_\delta^{T-\delta}\left|\tilde{u}_{\alpha_j}(t+\theta)-\tilde{u}_{\alpha_j}(t)\right|^2_{D(A)'}dt \\
&\le C\delta, \quad \widetilde{E}\left(\int_0^T\left\|\tilde{v}_{\alpha_j}(s)\right\|^2 ds\right) \\
&\le C_p, \quad \widetilde{E}\sup_{0\le s\le T}\left\|\tilde{v}_{\alpha_j}(s)\right\|^2 + 4\nu\beta\widetilde{E}\int_0^T\left\|\tilde{v}_{\alpha_j}(s)\right\|^2 ds \le C_1,
\end{aligned} \tag{6.27}$$

where $\widetilde{E}$ denote the mathematical expectation with respect to the probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$. Thus modulo the extraction of a subsequence denoted again $\tilde{u}_{\alpha_j}$ (with the corresponding $\tilde{v}_{\alpha_j}$), there exists two stochastic processes $\tilde{u}$, $\tilde{v}$ such that

$$\begin{aligned}
\tilde{u}_{\alpha_j} &\rightharpoonup \tilde{u} \quad \text{in } L^p\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^\infty(0,T;H)\right), \\
\tilde{u}_{\alpha_j} &\rightharpoonup \tilde{u} \quad \text{in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;V)\right), \\
\tilde{v}_{\alpha_j} &\rightharpoonup \tilde{v} \quad \text{in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;V)\right),
\end{aligned} \tag{6.28}$$

$$\begin{aligned}
&\widetilde{E}\sup_{0\le s\le T}|\tilde{u}(s)|^p \le C_p, \qquad \widetilde{E}\int_0^T\|\tilde{u}(s)\|_V^2 ds \le C, \\
&\widetilde{E}\sup_{0\le\theta\le\delta}\int_\delta^{T-\delta}|\tilde{u}(t+\theta)-\tilde{u}(t)|^2_{D(A)'}dt \le C\delta.
\end{aligned} \tag{6.29}$$

By (6.21), estimate (6.26), and Vitali's theorem, we have

$$\tilde{u}_{\alpha_j} \longrightarrow \tilde{u} \quad \text{in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;H)\right). \tag{6.30}$$

Thus modulo the extraction of a new subsequence and almost every (ω, t) with respect to the measure $d\overline{P}\otimes dt$

$$\tilde{u}_{\alpha_j} \longrightarrow \tilde{u} \quad \text{in } H. \tag{6.31}$$

Taking into account (6.30) and the Lipschitz condition on F, we have

$$\int_0^t F\left(s, \widetilde{u}_{\alpha_j}(s)\right) ds \longrightarrow \int_0^t F(s, \widetilde{u}(s)) ds \quad \text{in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;H)\right). \tag{6.32}$$

Arguing as in [5], we can prove that

$$\begin{aligned} &\int_0^t G\left(s, \widetilde{u}_{\alpha_j}(s)\right) \, d\widetilde{W}_{\alpha_j}(s) \\ &\quad \longrightarrow \textstyle\int_0^t G(s, \widetilde{u}(s)) \, d\widetilde{W}(s) \quad \text{in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^\infty(0,T;D(A)')\right) \text{ weakly star.} \end{aligned} \tag{6.33}$$

We also have

$$\widetilde{E}\int_0^T \left|\widetilde{v}_{\alpha_j}(t) - \widetilde{u}_{\alpha_j}(t)\right|^2 \, dt = \alpha_j^2 \widetilde{E}\int_0^T \alpha_j^2 \left|A\widetilde{u}_{\alpha_j}(t)\right|^2 dt. \tag{6.34}$$

We then deduce that

$$\widetilde{v}_{\alpha_j} \longrightarrow \widetilde{u} \quad \text{in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;H)\right), \tag{6.35}$$

since by the estimate (6.27), we have

$$\widetilde{E}\int_0^T \alpha_j^2 \left|A\widetilde{u}_{\alpha_j}(t)\right|^2 \, dt \text{ is bounded uniformly in } \alpha_j. \tag{6.36}$$

From (6.28) and (6.35), we have $\widetilde{v}(t) = \widetilde{u}(t)$ a.e. in $\overline{\omega} \times [0,T]$.

We are going to prove that

$$\int_0^t B\left(\widetilde{u}_{\alpha_j}(s), \widetilde{v}_{\alpha_j}(s)\right) ds \rightharpoonup \int_0^t B(\widetilde{u}(s), \widetilde{u}(s)) ds \quad \text{in } L^2\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}; L^2(0,T;D(A)')\right). \tag{6.37}$$

Indeed, let $\Phi \in \mathcal{V}$. From (2.5), (2.7), and (2.9), we have

$$\begin{aligned} &\int_0^t \left\langle B\left(\widetilde{u}_{\alpha_j}(s), \widetilde{v}_{\alpha_j}(s)\right), \Phi\right\rangle_{D(A)'} - \langle B(\widetilde{u}(s), \widetilde{u}(s)), \Phi\rangle_{D(A)'} ds \\ &\quad = \int_0^t \left\langle B\left(\widetilde{u}_{\alpha_j}(s) - \widetilde{u}(s), \widetilde{v}_{\alpha_j}(s)\right), \Phi\right\rangle_{D(A)'} ds + \int_0^t \left\langle B\left(\widetilde{u}(s), \widetilde{v}_{\alpha_j}(s) - \widetilde{u}(s)\right), \Phi\right\rangle_{D(A)'} ds \\ &\quad = \int_0^t \left\langle B\left(\widetilde{u}_{\alpha_j}(s) - u(s), \widetilde{v}_{\alpha_j}(s)\right), \Phi\right\rangle_{D(\mathrm{A})'} ds - \int_0^t \left(B(\widetilde{u}(s), \Phi), \widetilde{v}_{\alpha_j}(s) - \widetilde{u}(s)\right) ds \\ &\quad \leq C\int_0^t \left|\widetilde{u}_{\alpha_j}(s) - \widetilde{u}(s)\right| \left\|\widetilde{v}_{\alpha_j}(s)\right\| |A\Phi| \, ds + C\int_0^t \|\widetilde{u}(s)\| |A\Phi| \left|\widetilde{v}_{\alpha_j}(s) - \widetilde{u}(s)\right| ds. \end{aligned} \tag{6.38}$$

By Hölder's inequality

$$
\begin{aligned}
&\widetilde{E}\left(\int_0^t\left\langle B\left(\widetilde{u}_{\alpha_j}(s),\widetilde{v}_{\alpha_j}(s)\right),\Phi\right\rangle_{D(A)'}-\langle B(\widetilde{u}(s),\widetilde{u}(s)),\Phi\rangle_{D(A)'}ds\right)\\
&\quad\le C|A\Phi|\left(\widetilde{E}\int_0^t\left|\widetilde{u}_{\alpha_j}(s)-\widetilde{u}(s)\right|^2\,ds\right)^{1/2}\left(\widetilde{E}\int_0^t\left\|\widetilde{v}_{\alpha_j}(s)\right\|^2ds\right)^{1/2}\\
&\qquad+C|A\Phi|\left(\widetilde{E}\int_0^t\|\widetilde{u}(s)\|^2ds\right)^{1/2}\left(\widetilde{E}\int_0^t\left|\widetilde{v}_{\alpha_j}(s)-\widetilde{u}(s)\right|^2ds\right)^{1/2}.
\end{aligned}
\tag{6.39}
$$

It then follows from (6.30), (6.35), and (6.39) that

$$
\int_0^t B\left(\widetilde{u}_{\alpha_j}(s),\widetilde{v}_{\alpha_j}(s)\right)ds\rightharpoonup\int_0^t B(\widetilde{u}(s),\widetilde{u}(s))ds\quad\text{in } L^2\left(\overline{\Omega},\overline{\mathcal{F}},\overline{P};L^2\left(0,T;D(A)'\right)\right).
\tag{6.40}
$$

Collect all the convergence results and pass to the limit in (6.23) to obtain

$$
\begin{aligned}
&(\widetilde{u}(t),\Phi)+\nu\int_0^t(\widetilde{u}(s),A\Phi)ds+\int_0^t\langle B(\widetilde{u}(s),\widetilde{u}(s)),\Phi\rangle_{D(A)'}ds\\
&\quad=(u_0,\Phi)+\int_0^t(F(s,\widetilde{u}(s)),\Phi)ds+\int_0^t(G(s,\widetilde{u}(s)),\Phi)d\widetilde{W}(s).
\end{aligned}
\tag{6.41}
$$

This completes the proof of Theorem 6.5. □

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